18.453 Lecture 2

Plan

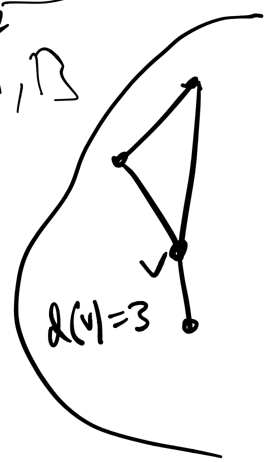
- Bipartite matchings
- König's theorem
  (vertex covers)
- Augmenting paths

5 min break → algorithm
- Hall's theorem

Bipartite matching

Recall from Tues: Graph $G = (V, E)$
vertices ↑ edges

degree $d(v)$ is # of edges incident to v.



More terms:

If $e = (a,b) \in E$, say e incident to a,b.

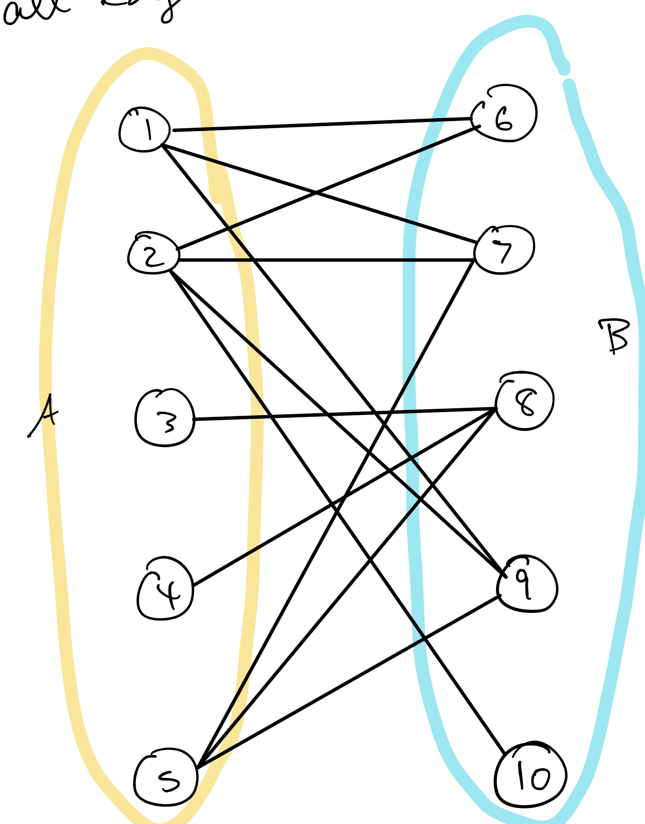
or a,b endpoints of e.

Def: Bipartite

called "bipartition"

G bipartite if V has partition A, B s.t. all edges between A & B.

Ex:



Fact:

G bipartite $\Leftrightarrow$ no odd cycles.

Def: Matchings

Recall <u>matching</u>: collection $M \subseteq E$ of disjoint edges.

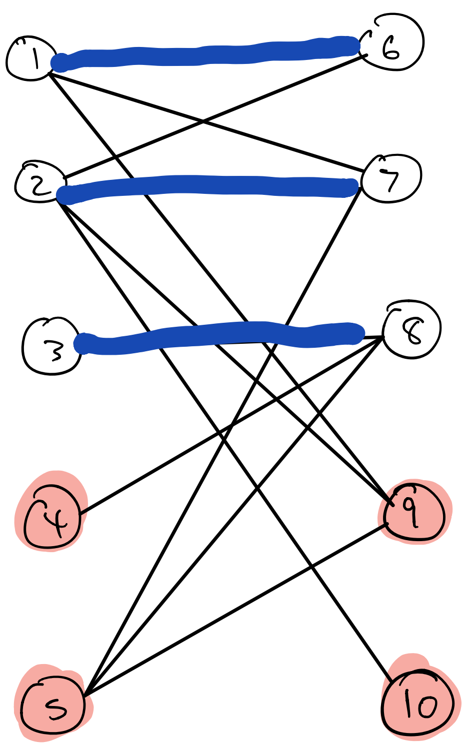
Say v <u>exposed</u> no edge in M incident to v.

M <u>perfect</u> if no exposed vertices.

Ex.

M in blue

Exposed red



Matchings useful!

<u>Problem 1: Cardinality perfect matching</u>

find matching M of maximum size.

Problem 2: Minimum weight perfect matching Given costs c_{ij} for all edges $(i,j) \in E$, find a perfect matching of minimum cost, where

$$\text{cost} = c(M) := \sum_{(i,j) \in M} c_{ij}.$$

Today we look at **Problem 1.**

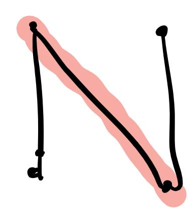
König's theorem

Before building algorithms, how's one certify that a matching is optimal (largest possible)?

Use obstruction to larger matching.

Duality again!

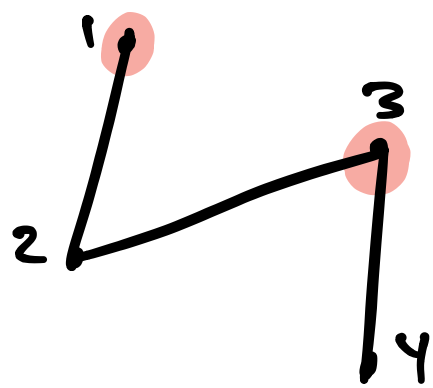
Def: Vertex Cover:



Set C of vertices is

a vertex cover for G if every edge $e \in E$ is incident to some $c \in C$.

Ex.



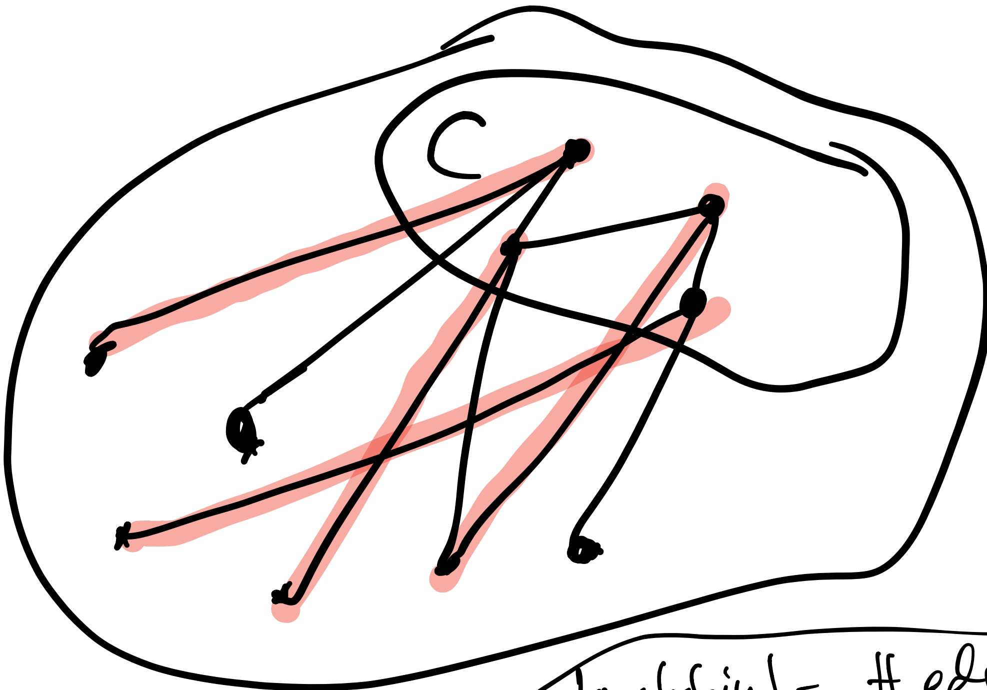
1, 3 form vertex cover

1, 4 do not.

Claim "weak duality"

$$|\text{any matching } M| \leq |\text{any vertex cover } C|$$

why? C contains at least one endpoint of every edge in M; but edges in M disjoint.



M

$|\text{matching}| = \#\text{edges}$

vertex covers are <u>only</u> obstructions for bipartite matching! strong duality.

<u>Theorem</u> (König 1931).

For any bipartite graph,

max |matching| = min |vertex cover|.

"min-max"

we'll prove this algorithmically.

Augmenting Paths Algorithm

outputs matching M, cover C with

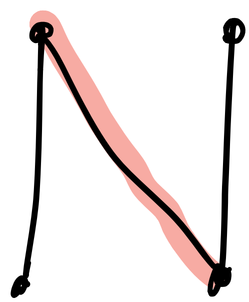
$$|M| = |C|;$$

by weak duality, $|M'| \le |C'|$

they must be max/min, respectively.

Note: greedy algorithm

won't work.



Def: Alternating path w.r.t. M

A path in G that alternates b/w edges in M and E − M.

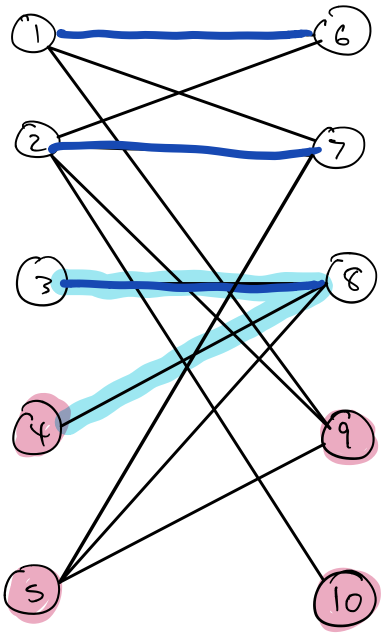
Def: Augmenting path w.r.t M

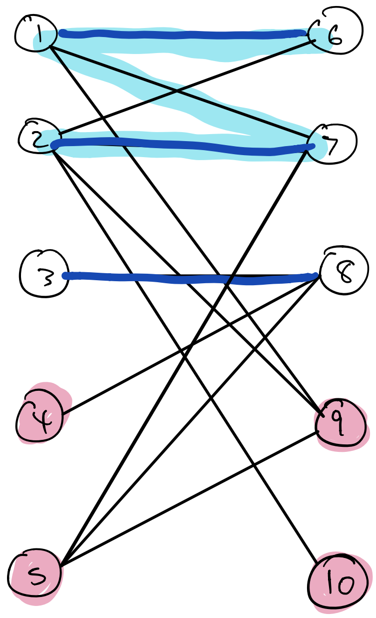
An alternating path whose first & last vertices are exposed.

alternating

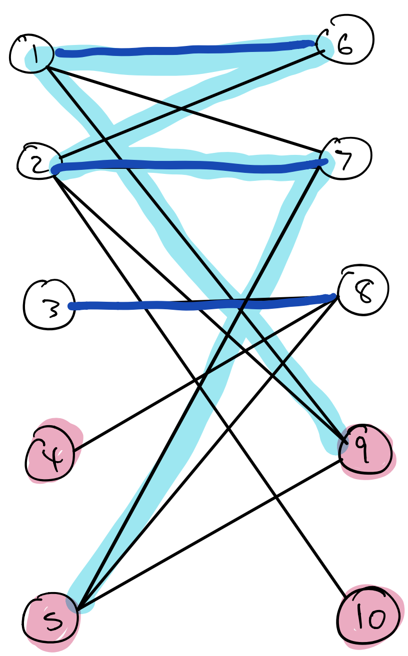
alternating

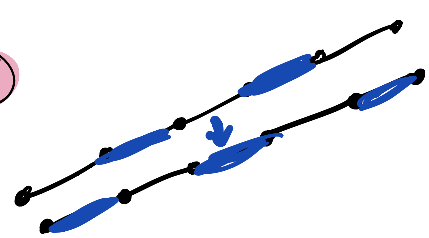
M





augmenting





Interesting properties

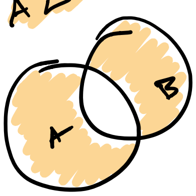
let P aug. path.

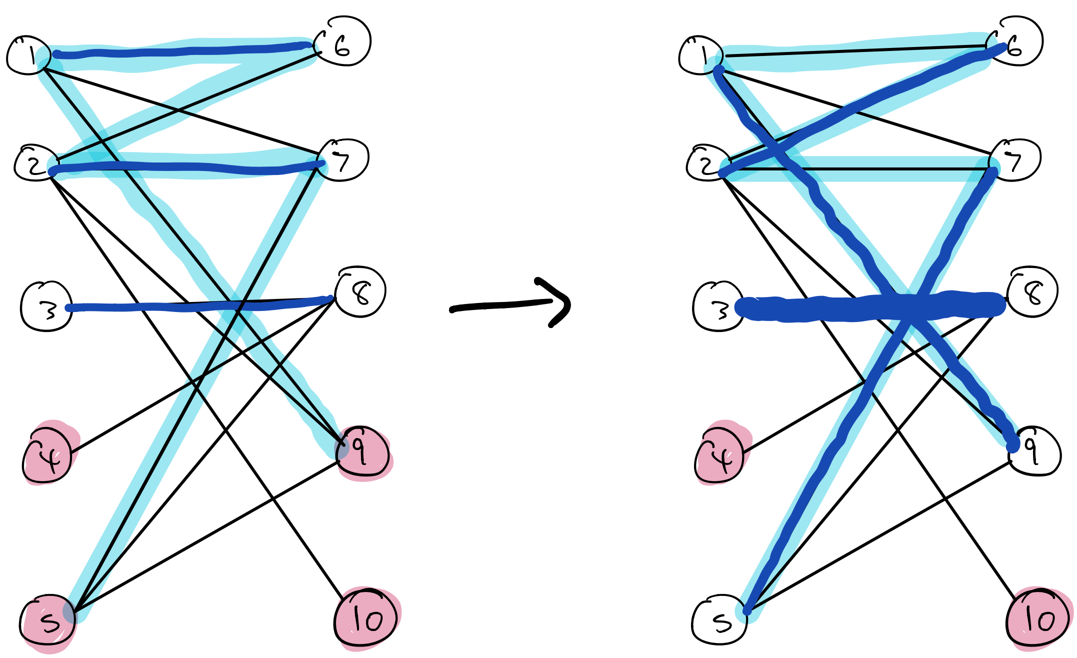
1. If P has k edges in M, has
   has k+1 edges not in M.

2. P's endpoints are on opposite sides (parity)

3. "Switch" edges in P: replace M by symmetric difference $M' = M \,\Delta\, P$ to obtain matching M' with one more edge.

$A \Delta B$





3   4

Equiv: replace edges in $P \cap M$ by edges in $P \setminus M$.

Say we have augmented M along P.

Thm: Matching M maximum
$\iff$ there are no augmenting paths w.r.t. M.

Proof: By contradiction.

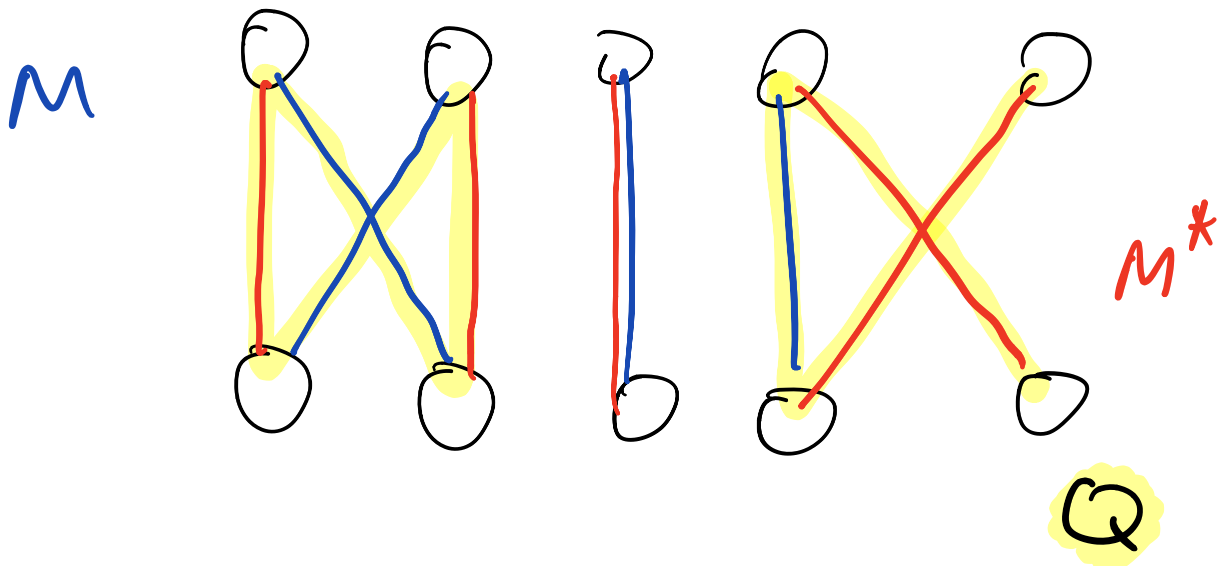
$\Rightarrow$ we already showed:

(if there is augmenting path, then exists bigger matching. ※

$\Leftarrow$ Assume M not maximum. Then let M^* be maximum, so $|M^*| > |M|$.

Let $Q = M \Delta M^*$

$= (M - M^*) \cup (M^* - M)$.



Then:

(1) Q has more edges from M^* than M

$|M^*| = |M \cap M^*| + |M^* - M|$

$|M| = |M \cap M^*| + |M - M^*|$

(2) Every vertex $v \in G$ adjacent to ≤ 1 EDGE in $M \cap Q$, ≤ 1 in $M^* \cap Q$.

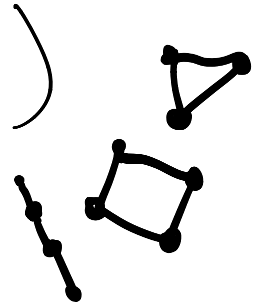
b/c M, M^* matchings.

(3) Q partitioned into vertex disjoint paths, cycles that alternate between M, M^*.

((2) $\Rightarrow$ every vertex in Q has deg 1 or 2.

$\Rightarrow$ Q decomp. into paths & cycles

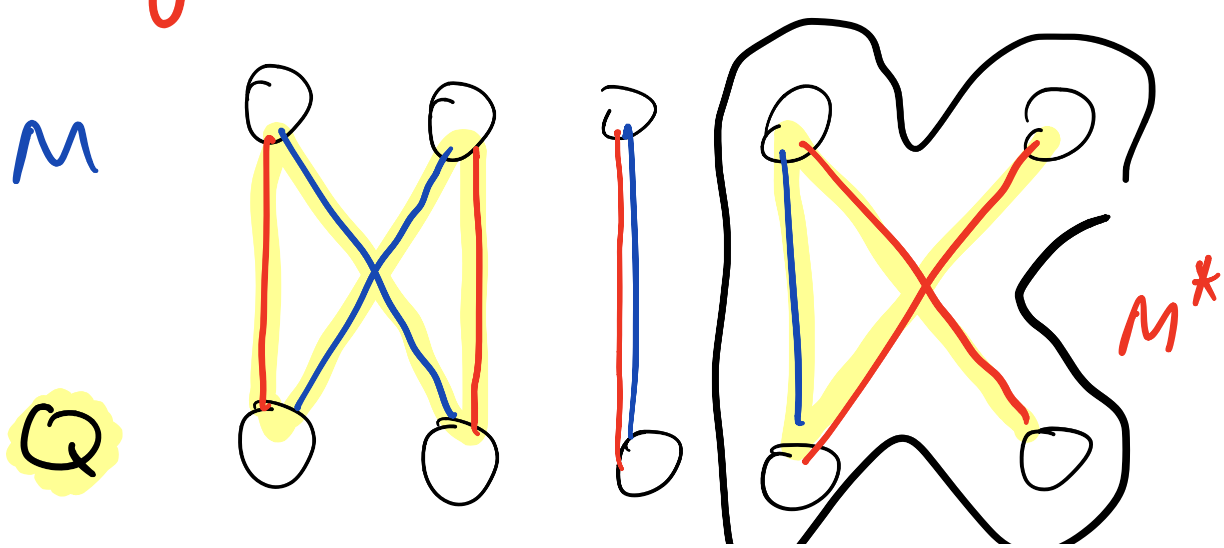
(2) $\Rightarrow$ paths & cycles alternate.)



4 Must be path in CQ with more edges from M^* than from M.

(b/c cycles are evenly split, and 1).

But this path is augmenting! Contradiction. $\square$

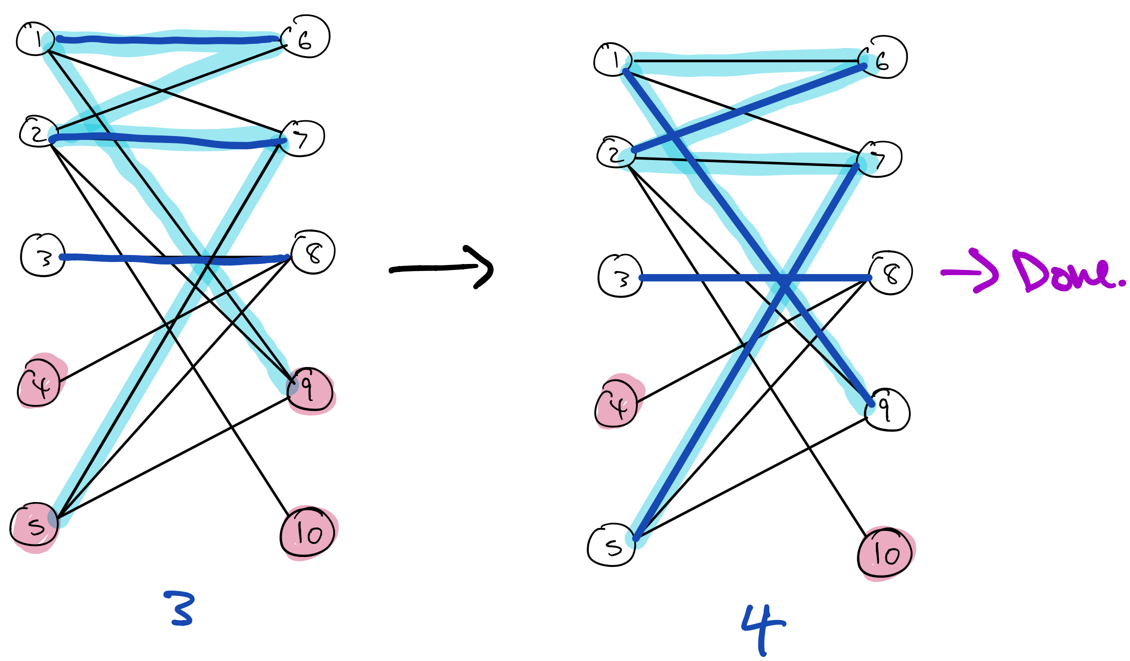


augmenting path for M.

Alg: Augmenting paths.

- Begin with any matching M.
- Find augmenting path P wrt M, augment M along P.
- Stop when no more augmenting paths.

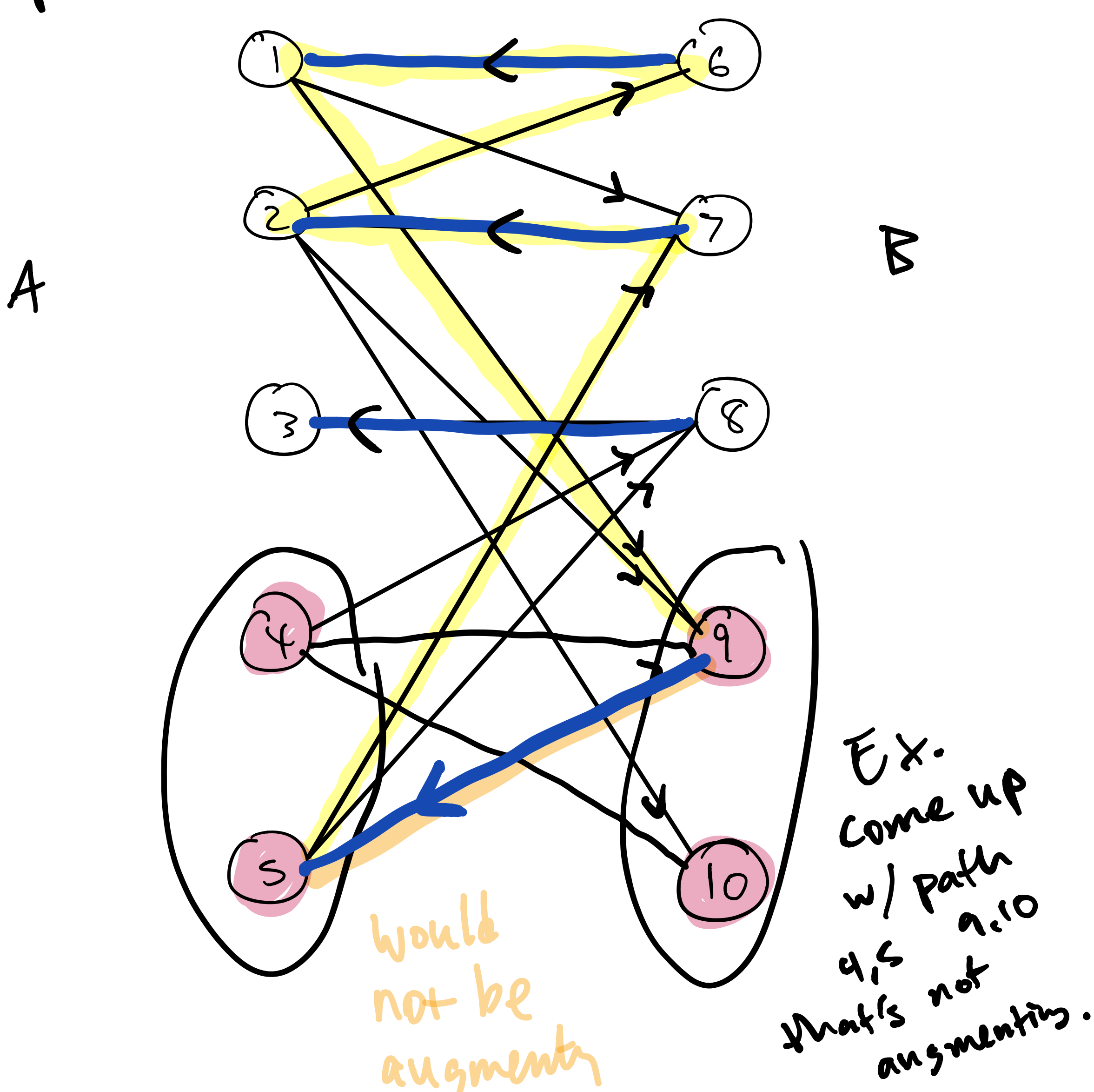
Ex:



Terminates after $\leq |M| \leq \frac{|V|}{2}$ augmentations.

BUT HOW TO FIND THE PATHS?

Reduces to finding path in a directed graph.

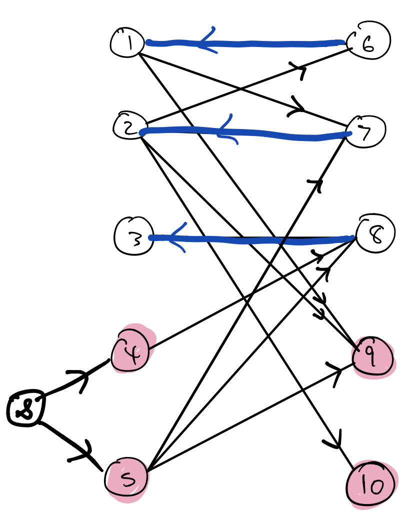


Direct e $A \to B$ if $e \in M$, $B \to A$ else.

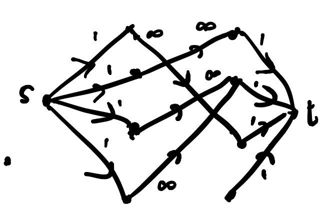
Augmenting path is precisely a directed path from <u>exposed vertex in A</u> to <u>exposed vertex in B</u>.

suggests to use depth-first search. (DFS)

Subroutine for Aug paths:



* direct graph as above.
* attach vertex s to exposed vertices in A
* do DFS until hit exposed vertex in B.
* Trace back path.



Takes $O(|E|)$ time to find augmenting path in G.

$\exists c$ s.t. time $\leq c|V||E|$.

(repeat $\frac{|V|}{2}$ times)

Thus, complexity is $O(|V||E|)$.

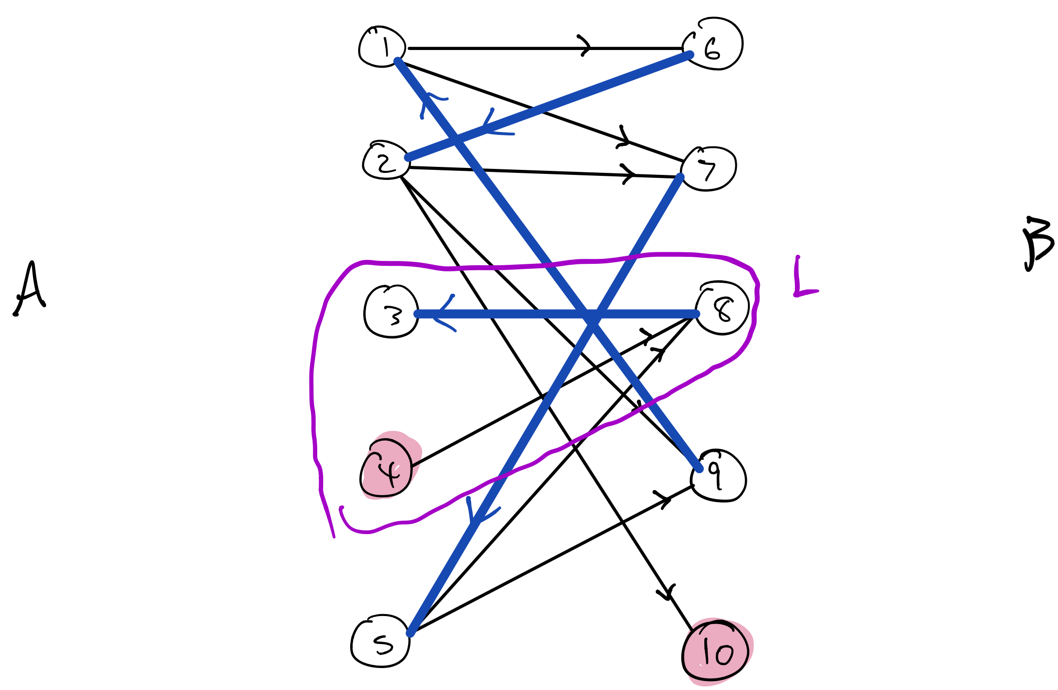
possible to get $O(\sqrt{n}\, m)$; Hopcroft-Karp.

$\sqrt{|V|}\,|E|$

Vertex Covers

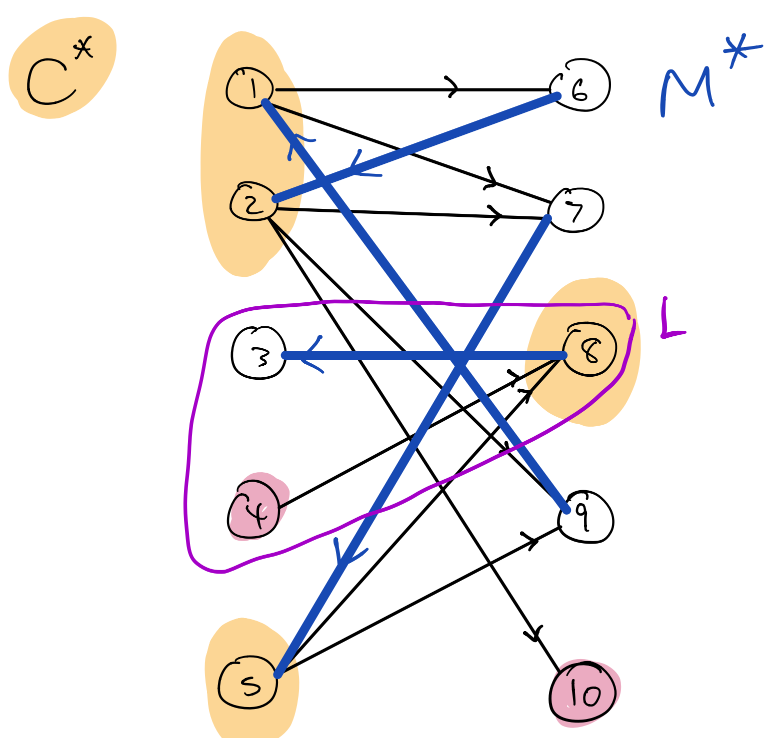
If no augmenting path wrt M, aug. path subroutine outputs a vertex cover. How?

Let L be set of vertices reachable by directed path from exposed vert in A.



Claim: When the algorithm terminates, $C^* = (\underline{A} - L) \cup (B \cap L)$'s a vertex cover & $|C^*| = |M^*|$

matching returned by alg.

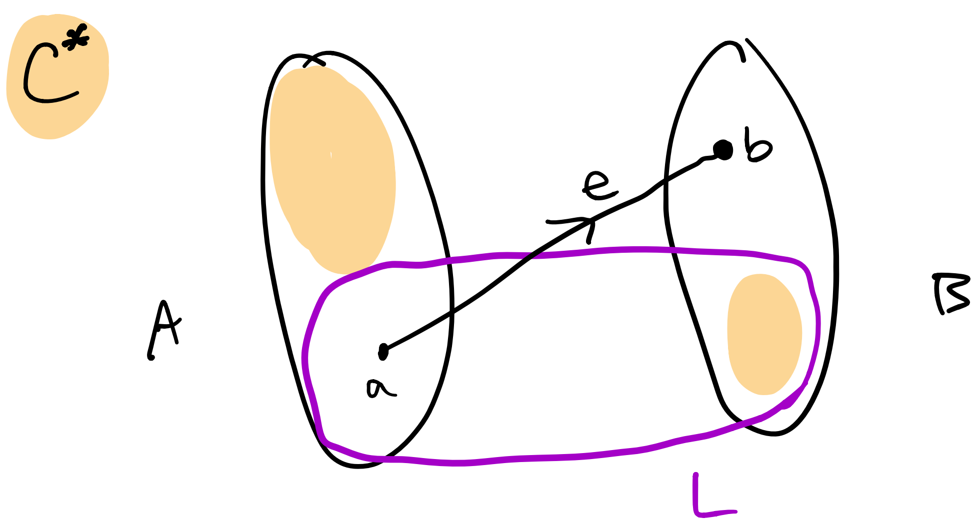


Corollary: König's Theorem

$\max |\text{matching}| = \min |\text{vertex cover}|$

Proof of Claim: First show C^* is cover. Assume not.

- Then exists $e = (a,b) \in E$ with $a \in A \cap L$, $b \in (B - L)$



- $e \notin M$. 1) if $e \in M$ then a is not exposed.

2) if $e \in M$, then

e is only $B \to A$ edge to a

$\Rightarrow a$ not reachable;

contradicts $a \in L$.

- Thus, b reachable; contradicts $b \notin L$.

C^* is cover.

$\downarrow$

$\square$

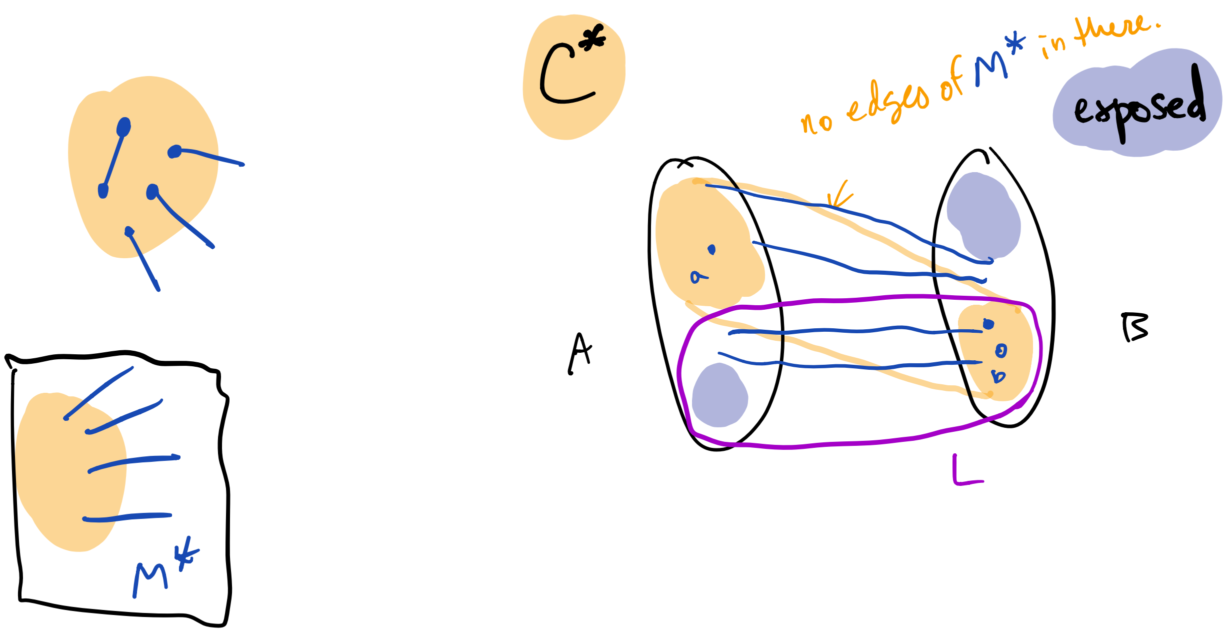
Now to prove $|M^*| = |C^*|$.

* Enough to show $\underset{\#\text{verts}}{|C^*|} \leq \underset{\#\text{edges}}{|M^*|}$.

weak duality: $|M^*| \leq |C^*|$.

To prove,* observe:

- No vertex in $A-L$ exposed
  (or else would be reachable, i.e. in L).
- No vertex in $B \cap L$ exposed
  (else algorithm wouldn't have terminated)
- No edge of M b/w $a \in A-L$, $b \in B \cap L$

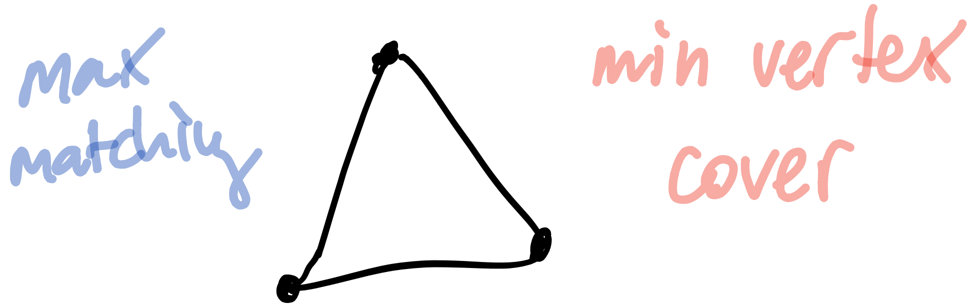


Conclude: every vertex in C^* matched, and no edge of M fully within C^*.
hence $|C^*| \le |M^*|$. $\square$

What about general graphs?

Still have weak duality,

but not strong duality:

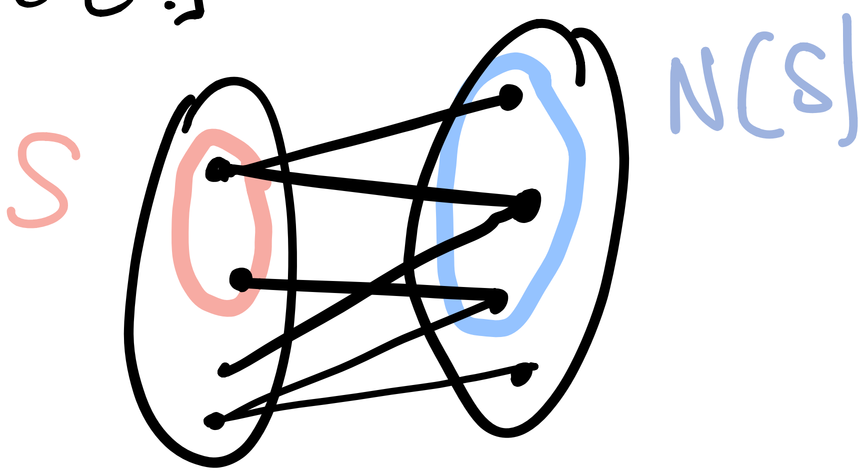


Need Tutte's theorem from lec 1; we'll discuss later.

Hall's theorem

Hall's theorem is another "duality" characterization of the existence of a perfect matching.

Def: neighborhood $N(S)$ of a set S is $\{b \text{ s.t. } \exists a \in S \text{ w/ } (a, b) \in E.\}$

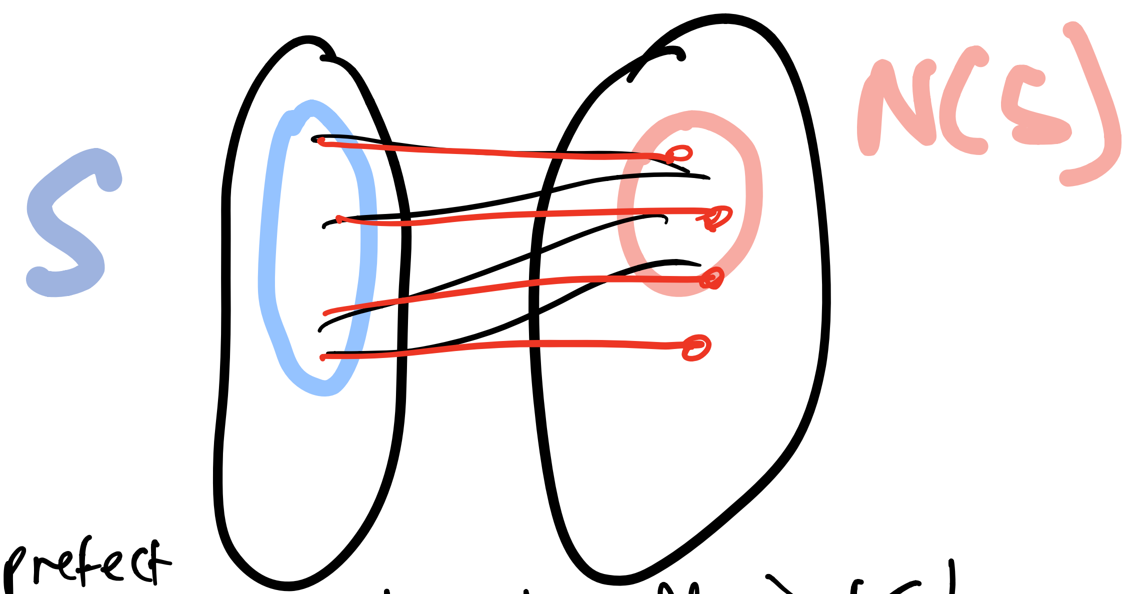


THM (Hall) Bipartite G with bipartition A, B has perfect matching

$$\Longleftrightarrow$$

$$\forall S \subseteq A, \quad |N(S)| \geq |S|.$$

Clearly, $|N(S)| \geq |S|$ necessary;



(if prefect matching, then $|N(S)| \geq |S|$.

i.e. $|N(S)|<|S|$ obstructs p.m.'s; weak duality. Hall says strong duality here also.

Hall's follows from König's theorem.
(relate S, $N(S)$ to vertex cover).

See exercises in source.

Ex 1-9 is to prove Hall's theorem from König's theorem.